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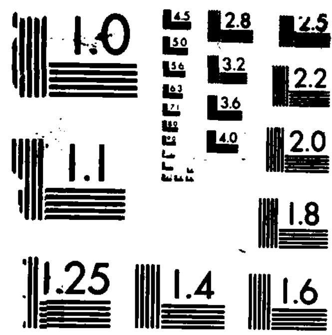

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Incomplete Lipschitz-Hankel Integrals of Bessel Functions

ALLEN R. MILLER

Computer-Aided Design/Computer-Aided Manufacturing
Engineering Services Division

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Various representations for incomplete Lipschitz-Hankel integrals of Bessel functions have been given in terms of Kampé de Fériet double hypergeometric functions. Reduction formulas for the double series employed have been given in some cases.

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CONTENTS

INCOMPLETE LIPSCHITZ-HANKEL INTEGRALS OF BESSEL FUNCTIONS

INTRODUCTION

The general incomplete Lipschitz-Hankel Integral of Bessel Functions of the first kind is defined by

$$J_{e_{\mu,\nu}}(a, z) \equiv \int_0^z e^{at} t^{\mu} J_{\nu}(t)\, dt \tag{1}$$

Here the symbol e denotes the presence of the exponential function, and μ, ν may be complex numbers. Analogously, we may define integrals that contain the functions sin (at) and cos (at) in place of exp (at):

$$J_{s_{\mu,\nu}}(a, z) \equiv \int_0^z \sin\,(\mathrm{at})\, t^{\mu} J_{\nu}(t)\, dt \tag{2}$$

$$J_{c_{\mu,\nu}}(a, z) \equiv \int_0^z \cos\,(\mathrm{at})\, t^{\mu} J_{\nu}(t)\, dt \tag{3}$$

To assure convergence of these integrals, it is necessary that $Re\,(1 + \mu + \nu) > 0$. When $\mu = \nu$ we shall write, for example,

$$J_{e_{\mu,\mu}}(a, z) \equiv J_{e_{\mu}}(a, z) \tag{4}$$

We shall also define integrals of modified Bessel functions $I_{\nu}(t)$ or other cylindrical functions $C(t)$ by simply replacing J by I or C in the above definitions. In addition, we define $J^{+} \equiv J$, $J^{-} \equiv I$.

In Ref. 1 it is shown for the Bessel function of imaginary argument or MacDonald function K_0 that

$$K_{e_0}(a, z) = z\, K_0(z)\, A(a, z) + z^2 K_1(z)\, B(a, z)$$

where

$$A(a, z) \equiv L\left[\frac{1}{2}, 1; \frac{1}{2}, \frac{3}{2}; \frac{a^2 z^2}{4}, \frac{z^2}{4}\right] + \frac{az}{2} Q\left[1, 1, 1; 1, 2, \frac{3}{2}; \frac{a^2 z^2}{4}, \frac{z^2}{4}\right]$$

$$B(a, z) \equiv L\left[\frac{1}{2}, 1; \frac{3}{2}, \frac{3}{2}; \frac{a^2 z^2}{4}, \frac{z^2}{4}\right] + \frac{az}{4} Q\left[1, 1, 1; 2, 2, \frac{3}{2}; \frac{a^2 z^2}{4}, \frac{z^2}{4}\right]$$

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Here L and Q are Kampé de Fériet double hypergeometric functions (defined below) of order three and four respectively. These functions are therefore non-Gaussian. Only members of the class of double Gaussian series of order two that consists of 34 distinct convergent forms have been given names [2, p. 54]. These 34 forms are sometimes referred to as Horn's list.

In this report we shall show that the functions L and Q may also be employed to give representations for Eqs. 1-4 for I and J. To this end we recall the definitions of the Kampé de Fériet functions L and Q:

$$Q[\alpha, \beta, \gamma; \mu, \nu, \lambda; x, y] \equiv F\,{}^{0:2;1}_{2:1;0}\left[\begin{matrix} - : \alpha, \beta \,;\, \gamma \,; \\ \mu, \nu : \quad \gamma \,;\, - \,; \end{matrix}\, x, y\right]$$

$$L[\alpha, \beta; \gamma, \delta; x, y] \equiv Q[\alpha, \lambda, \beta; \gamma, \delta, \lambda; x, y] \qquad |x| < \infty,\ |y| < \infty$$

We shall also introduce the third order function

$$N[\alpha; \beta, \gamma, \delta; x, y] \equiv F\,{}^{1:0;0}_{1:1;1}\left[\begin{matrix} \alpha : - \,;\, - \,; \\ \beta : \gamma \,;\, \delta \,; \end{matrix}\, x, y\right] \qquad |x| < \infty,\ |y| < \infty$$

REPRESENTATIONS FOR $\mathbf{J}^{\pm}_{e_{\mu,\nu}}(\mathbf{a}, \mathbf{z})$, $\mathbf{J}^{\pm}_{c_{\mu,\nu}}(\mathbf{a}, \mathbf{z})$, $\mathbf{J}^{\pm}_{s_{\mu,\nu}}(\mathbf{a}, \mathbf{z})$

Since

$$J_\nu^{\pm}(t) = \frac{(t/2)^\nu}{\Gamma(1+\nu)}\, {}_0F_1[-; 1+\nu; \mp t^2/4]$$

we easily find that

$$e^{at}\, t^\mu J_\nu^{\pm}(t) = \frac{1}{2^\nu \Gamma(1+\nu)} \sum_{n=0}^{\infty} \frac{a^n}{n!} \sum_{m=0}^{\infty} \frac{(\mp 1)^m\, t^{\mu+\nu+2m+n}}{2^{2m}(1+\nu)_m\, m!}$$

Now assuming that $Re(1+\mu+\nu) > 0$ we obtain, on integrating term by term with respect to t,

$$J^{\pm}_{e_{\mu,\nu}}(a, z) = \frac{z^{1+\mu+\nu}}{2^\nu \Gamma(1+\nu)} \sum_{m,n=0}^{\infty} \frac{(az)^n}{n!} \frac{(\mp z^2/4)^m}{m!} \frac{1}{(1+\nu)_m (1+\mu+\nu+2m+n)} \tag{5}$$

Substituting

$$\frac{1}{1+\mu+\nu+2m+n} = \frac{1}{1+\mu+\nu} \frac{\left(\frac{1+\mu+\nu}{2}\right)_m}{\left(\frac{3+\mu+\nu}{2}\right)_m} \frac{(1+\mu+\nu+2m)_n}{(2+\mu+\nu+2m)_n}$$

into Eq. 5 then gives

$$J^{\pm}_{e_{\mu,\nu}}(a, z) = \frac{z^{1+\mu+\nu}}{2^{\nu}(1+\mu+\nu)\,\Gamma(1+\nu)} \sum_{m=0}^{\infty} \frac{\left(\frac{1+\mu+\nu}{2}\right)_m}{\left(\frac{3+\mu+\nu}{2}\right)_m (1+\nu)_m} \frac{(\mp z^2/4)^m}{m!}$$

$$\cdot\, {}_1F_1[1+\mu+\nu+2m;\, 2+\mu+\nu+2m;\, az] \tag{6}$$

Now using Kummer's first theorem

$${}_1F_1[a;\, c;\, z] = e^z\, {}_1F_1\,[c-a;\, c;\, -z]$$

we obtain from Eq. 6

$$J^{\pm}_{e_{\mu,\nu}}(a, z) = \frac{z^{1+\mu+\nu}e^{az}}{2^{\nu}(1+\mu+\nu)\Gamma(1+\nu)} \sum_{m=0}^{\infty} \frac{\left(\frac{1+\mu+\nu}{2}\right)_m}{\left(\frac{3+\mu+\nu}{2}\right)_m (1+\nu)_m} \frac{(\mp z^2/4)^m}{m!}$$

$$\cdot\, {}_1F_1\,[1;\, 2+\mu+\nu+2m;\, -az] \tag{7}$$

Since

$$\frac{1}{(2+\mu+\nu+2m)_n} = \frac{2^{2m}\left(\frac{2+\mu+\nu}{2}\right)_m \left(\frac{3+\mu+\nu}{2}\right)_m}{(2+\mu+\nu)_{2m+n}}$$

we obtain from Eq. 7

$$J^{\pm}_{e_{\mu,\nu}}(a, z) = \frac{z^{1+\mu+\nu}\,e^{az}}{2^{\nu}(1+\mu+\nu)\Gamma(1+\nu)} \sum_{m,n=0}^{\infty} \frac{(\mp z^2)^m}{m!} \frac{(-az)^n}{n!}$$

$$\cdot\, \frac{(1)_n \left(\frac{1+\mu+\nu}{2}\right)_m \left(\frac{2+\mu+\nu}{2}\right)_m}{(1+\nu)_m\,(2+\mu+\nu)_{2m+n}} \tag{8}$$

Finally, noting that for any α

$$(2+\alpha)_{2m+2n} = 2^{2m}2^{2n}\left(\frac{2+\alpha}{2}\right)_{m+n}\left(\frac{3+\alpha}{2}\right)_{m+n}$$

$$(2+\alpha)_{2m+2n+1} = (2+\alpha)2^{2m}2^{2n}\left(\frac{3+\alpha}{2}\right)_{m+n}\left(\frac{4+\alpha}{2}\right)_{m+n}$$

we obtain from Eq. 8 and the definition of Q given earlier

$$J^{\pm}_{e_{\mu,\nu}}(a,z) = \frac{z^{1+\mu+\nu}\,e^{az}}{2^{\nu}(1+\mu+\nu)\Gamma(1+\nu)} \tag{9}$$

$$\cdot\Bigg\{Q\Big[\frac{1+\mu+\nu}{2},\frac{2+\mu+\nu}{2},1;\frac{2+\mu+\nu}{2},\frac{3+\mu+\nu}{2},1+\nu;\frac{\mp z^2}{4},\frac{a^2z^2}{4}\Big]$$

$$-\frac{az}{2+\mu+\nu}\,Q\Big[\frac{1+\mu+\nu}{2},\frac{2+\mu+\nu}{2},1;\frac{3+\mu+\nu}{2},\frac{4+\mu+\nu}{2},1+\nu;\frac{\mp z^2}{4},\frac{a^2z^2}{4}\Big]\Bigg\}$$

On letting $\mu = \nu$ in Eq. 9 we have

$$J^{\pm}_{e_{\mu}}(a,z) = \frac{z(z^2/2)^{\mu}\,e^{az}}{(1+2\mu)\Gamma(1+\mu)}$$

$$\cdot\Bigg\{L\Big[\frac{1}{2}+\mu,1;1+\mu,\frac{3}{2}+\mu;\frac{\mp z^2}{4},\frac{a^2z^2}{4}\Big]$$

$$-\frac{az}{2(1+\mu)}\,L\Big[\frac{1}{2}+\mu,1;\frac{3}{2}+\mu,2+\mu;\frac{\mp z^2}{4},\frac{a^2z^2}{4}\Big]\Bigg\}$$

In addition we may use Eq. 5 and the definition of N to obtain

$$J^{\pm}_{e_{\mu,\nu}}(a,z) = \frac{z^{1+\mu+\nu}}{2^{\nu}\Gamma(1+\nu)}$$

$$\cdot\Bigg\{\frac{1}{1+\mu+\nu}\,N\Big[\frac{1+\mu+\nu}{2};\frac{3+\mu+\nu}{2},1+\nu,\frac{1}{2};\frac{\mp z^2}{4},\frac{a^2z^2}{4}\Big]$$

$$+\frac{az}{2+\mu+\nu}\,N\Big[\frac{2+\mu+\nu}{2};\frac{4+\mu+\nu}{2},1+\nu,\frac{3}{2};\frac{\mp z^2}{4},\frac{a^2z^2}{4}\Big]\Bigg\} \tag{10}$$

For brevity we shall define the following parameter lists ∇_j:

$$\nabla_1 \equiv \frac{1+\mu+\nu}{2}, \frac{2+\mu+\nu}{2}, 1; \frac{2+\mu+\nu}{2}, \frac{3+\mu+\nu}{2}, 1+\nu$$

$$\nabla_2 \equiv \frac{1+\mu+\nu}{2}, \frac{2+\mu+\nu}{2}, 1; \frac{3+\mu+\nu}{2}, \frac{4+\mu+\nu}{2}, 1+\nu$$

$$\nabla_3 \equiv \frac{1+\mu+\nu}{2}; \frac{3+\mu+\nu}{2}, 1+\nu, \frac{1}{2}$$

$$\nabla_4 \equiv \frac{2+\mu+\nu}{2}; \frac{4+\mu+\nu}{2}, 1+\nu, \frac{3}{2}$$

$$\nabla_5 \equiv 1+\mu+\nu, \frac{1}{2}+\nu; 2+\mu+\nu, 1+2\nu$$

We may then obtain from Eqs. 9 and 10

$$J_{c_{\mu,\nu}}^{\pm}(a, z) = \frac{z^{1+\mu+\nu}}{2^\nu(1+\mu+\nu)\Gamma(1+\nu)} \left\{\cos(az)\, Q[\nabla_1; \frac{\mp z^2}{4}, \frac{-a^2z^2}{4}] + \frac{az}{2+\mu+\nu} \sin(az)\, Q[\nabla_2; \frac{\mp z^2}{4}, \frac{-a^2z^2}{4}]\right\}$$

$$= \frac{z^{1+\mu+\nu}}{2^\nu(1+\mu+\nu)\Gamma(1+\nu)} N[\nabla_3; \frac{\mp z^2}{4}, \frac{-a^2z^2}{4}] \tag{11}$$

$$J_{s_{\mu,\nu}}^{\pm}(a, z) = \frac{z^{1+\mu+\nu}}{2^\nu(1+\mu+\nu)\Gamma(1+\nu)} \left\{\sin(az)\, Q[\nabla_1; \frac{\mp z^2}{4}, \frac{-a^2z^2}{4}] - \frac{az}{2+\mu+\nu} \cos(az)\, Q[\nabla_2; \frac{\mp z^2}{4}, \frac{-a^2z^2}{4}]\right\}$$

$$= \frac{az^{2+\mu+\nu}}{2^\nu(2+\mu+\nu)\Gamma(1+\nu)} N[\nabla_4; \frac{\mp z^2}{4}, \frac{-a^2z^2}{4}]$$

And from these equations we obtain on letting $\mu = \nu$

$$J_{c_\mu}^{\pm}(a,z) = \frac{z^{1+2\mu}}{2^{\mu}(1+2\mu)\Gamma(1+\mu)} \left\{ \cos(az)\, L[\frac{1}{2}+\mu, 1; 1+\mu, \frac{3}{2}+\mu; \frac{\mp z^2}{4}, \frac{-a^2z^2}{4}] \right.$$

$$\left. + \frac{az}{2(1+\mu)} \sin(az)\, L[\frac{1}{2}+\mu, 1; \frac{3}{2}+\mu, 2+\mu; \frac{\mp z^2}{4}, \frac{-a^2z^2}{4}] \right\}$$

$$= \frac{z^{1+2\mu}}{2^{\mu}(1+2\mu)\Gamma(1+\mu)} N[\frac{1}{2}+\mu; \frac{3}{2}+\mu, 1+\mu, \frac{1}{2}; \frac{\mp z^2}{4}, \frac{-a^2z^2}{4}]$$

$$J_{s_\mu}^{\pm}(a,z) = \frac{z^{1+2\mu}}{2^{\mu}(1+2\mu)\Gamma(1+\mu)} \left\{ \sin(az)\, L[\frac{1}{2}+\mu, 1; 1+\mu, \frac{3}{2}+\mu; \frac{\mp z^2}{4}, \frac{-a^2z^2}{4}] \right.$$

$$\left. - \frac{az}{2(1+\mu)} \cos(az)\, L[\frac{1}{2}+\mu, 1; \frac{3}{2}+\mu, 2+\mu; \frac{\mp z^2}{4}, \frac{-a^2z^2}{4}] \right\}$$

$$= \frac{az^{2(1+\mu)}}{2^{1+\mu}(1+\mu)\Gamma(1+\mu)} N[1+\mu; 2+\mu, 1+\mu, \frac{3}{2}; \frac{\mp z^2}{4}, \frac{-a^2z^2}{4}]$$

Finally, noting that $I_\nu(z)$ may be represented by

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1+\nu)} e^{\pm z}\, {}_1F_1\, [\frac{1}{2}+\nu; 1+2\nu; \mp 2z]$$

we readily obtain

$$I_{e_{\mu,\nu}}(a,z) = \frac{z^{1+\mu+\nu}}{2^{\nu}(1+\mu+\nu)\Gamma(1+\nu)} F\begin{matrix}1:1;0\\1:1;0\end{matrix}\left[\begin{matrix}1+\mu+\nu: & 1/2+\nu; & -; \\ 2+\mu+\nu: & 1+2\nu; & -;\end{matrix}\ \pm 2z, (\_ \mp 1)z\right] \tag{13}$$

REDUCTION FORMULAS FOR L, N, Q

In some instances $J_{e_{\mu,\nu}}^{\pm}(a,z)$ may be expressed in terms of generalized hypergeometric functions provided that we know a reduction formula for one of L, N, or Q. By using Ref. 3, p. 55, Eqs. 19, 20, and 21 respectively we find

$$N[\alpha; \beta, \gamma, \gamma; x, -x] = {}_2F_5[\frac{\alpha}{2}, \frac{\alpha+1}{2}; \frac{\beta}{2}, \frac{\beta+1}{2}, \gamma, \frac{\gamma}{2}, \frac{\gamma+1}{2}; \frac{-x^2}{4}]$$

$$L[\alpha, \beta; \gamma, \delta; x, x] = {}_1F_2[\alpha \div \beta; \gamma, \delta; x]$$

$$L[\alpha, \alpha; \gamma, \delta; x, -x] = {}_1F_4[\alpha; \frac{\gamma}{2}, \frac{\gamma+1}{2}, \frac{\delta}{2}, \frac{\delta+1}{2}; \frac{x^2}{16}]$$

Using Ref. 2, p. 28. Eqs. 33 and 34 respectively we find

$$N[\alpha; \beta, \gamma, \delta; x, x] = {}_3F_4[\alpha, \frac{\gamma+\delta-1}{2}, \frac{\gamma+\delta}{2}; \beta, \gamma, \delta, \gamma+\delta-1; 4x] \tag{14}$$

$$Q[\frac{-1/2+\nu}{2}, \frac{1/2+\nu}{2}, 1; \alpha, \beta, 1+\nu; x, x] = {}_2F_3[\frac{3+2\nu}{4}, \frac{5+2\nu}{4}; \alpha, \beta, 1+\nu; x]$$

Employing Eqs. 9 and 13, we easily deduce

$$Q[\nabla_1; \frac{z^2}{4}, \frac{z^2}{4}] = \frac{1}{2}\{e^z\, {}_2F_2[\nabla_5; -2z] + e^{-z}\, {}_2F_2[\nabla_5; 2z]\}$$

$$Q[\nabla_2; \frac{z^2}{4}, \frac{z^2}{4}] = \frac{2+\mu+\nu}{2z}\{e^z\, {}_2F_2[\nabla_5; -2z] - e^{-z}\, {}_2F_2[\nabla_5; 2z]\}$$

And finally, using Eqs. 11 and 12 we find

$$Q[\nabla_1; \frac{-z^2}{4}, \frac{-z^2}{4}] = \cos z\; N[\nabla_3; \frac{-z^2}{4}, \frac{-z^2}{4}]$$

$$+ \frac{1+\mu+\nu}{2+\mu+\nu} z \sin z\; N[\nabla_4; \frac{-z^2}{4}, \frac{-z^2}{4}]$$

$$Q[\nabla_2; \frac{-z^2}{4}, \frac{-z^2}{4}] = (2+\mu+\nu)\frac{\sin z}{z}\; N[\nabla_3; \frac{-z^2}{4}, \frac{-z^2}{4}]$$

$$-(1+\mu+\nu)\cos z\; N[\nabla_4; \frac{-z^2}{4}, \frac{-z^2}{4}]$$

Replacing z by iz in these equations then gives

$$Q[\nabla_1; \frac{z^2}{4}, \frac{z^2}{4}] = \cosh z\; N[\nabla_3; \frac{z^2}{4}, \frac{z^2}{4}] - \frac{1+\mu+\nu}{2+\mu+\nu} z \sinh z\; N[\nabla_4; \frac{z^2}{4}, \frac{z^2}{4}]$$

$$Q[\nabla_2; \frac{z^2}{4}, \frac{z^2}{4}] = (2+\mu+\nu)\frac{\sinh z}{z}\; N[\nabla_3; \frac{z^2}{4}, \frac{z^2}{4}] - (1+\mu+\nu)\cosh z\; N[\nabla_4; \frac{z^2}{4}, \frac{z^2}{4}]$$

where, on using Eq. 14,

$$N[\nabla_3; \frac{z^2}{4}, \frac{z^2}{4}] = {}_3F_4[\frac{1+\mu+\nu}{2}, \frac{1/2+\nu}{2}, \frac{3/2+\nu}{2}; \frac{3+\mu+\nu}{2}, \frac{1}{2}+\nu, 1+\nu, \frac{1}{2}; z^2]$$

$$N[\nabla_4; \frac{z^2}{4}, \frac{z^2}{4}] = {}_3F_4[\frac{2+\mu+\nu}{2}, \frac{3/2+\nu}{2}, \frac{5/2+\nu}{2}; \frac{4+\mu+\nu}{2}, \frac{3}{2}+\nu, 1+\nu, \frac{3}{2}; z^2]$$

SUMMARY

Various representations for incomplete Lipschitz-Hankel integrals of Bessel functions have been given in terms of Kampé de Fériet double hypergeometric functions. Reduction formulas for the double series employed have been given in some cases.

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